

Thermal renormalization group for fermions, universality, and the chiral phase transition

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We formulate the thermal renormalization group, an implementation of the Wilsonian RG in the real-time (CTP) formulation of finite temperature field theory, for fermionic fields. Using a model with scalar and fermionic degrees of freedom which should describe the two-flavor chiral phase transition, we discuss the mechanism behind fermion decoupling and universality at second order transitions. It turns out that an effective mass-like term in the fermion propagator which is due to thermal fluctuations and does not break chiral symmetry is necessary for fermion decoupling to work. This situation is in contrast with the high-temperature limit, where the dominance of scalar over fermionic degrees of freedom is due to the different behavior of the distribution functions. The mass-like contribution is the leading thermal effect in the fermionic sector and is missed if a derivative expansion of the fermionic propagator is performed. We also discuss results on the phase transition of the model considered where we find good agreement with results from other methods.

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I. INTRODUCTION

Finite temperature field theory has become a mature part of theoretical particle physics over the past 25 years. Since the seminal work of Kirzhnits and Linde [1] on symmetry restoration at high temperatures there has been a strong theoretical interest in methods and results in the field. In recent years, the experimental quest for the high temperature state of strongly interacting matter, the quark-gluon plasma, has attracted additional interest. Nevertheless, there has a large number of open problems both on a phenomenological as well as on a fundamental level. These range from the equation of state of nuclear matter at high temperature and density (of interest e.g. in astrophysics) to questions about the approach of a quantum system to thermal equilibrium.

The most general approach to quantum field theory in a nontrivial “environment” (as specified by a density matrix at some initial time t_i) was proposed by Schwinger [2] and Keldysh [3] and is known under the name of the closed time-path (CTP) formulation of quantum field theory. This approach allows for the calculation of operator expectation values in quite general systems, including systems in thermal and chemical equilibrium. The special case of equilibrium physics has, because of its relative simplicity, attracted most of the attention in the field. It has also led to two (in principle equivalent) formulations, known as the Matsubara or imaginary time and the real-time formulation of thermal field theory (see e.g. [4,5]). As the names suggest, in the (more widely used) Matsubara approach one formulates the theory in Euclidean space (with a finite time interval), the price to pay for this being the necessity of analytically continuing all time arguments of Green functions at the end of the calculations. The real-time formulation, on the other hand, works in Minkowski space throughout and thus the need to do analytical continuations never arises.

As long as one is mainly interested in static quantities such as the free energy, etc., there is of course no need to consider time dependence. Since both formulations should yield the same answers for all quantities one will usually pick the technically simpler one—which most of the time will be the Matsubara approach. However, even in a calculation within a thermal system, time-dependent correlation functions carry important physical information to be interpreted within linear response theory [5]. Thus for example the decay of out-of-equilibrium fluctuations of a given system is governed by the “damping rate,” being connected to the imaginary part of the self energy of the corresponding fields. This quantity plays an important rôle in systems approaching the critical temperature of a second order phase transition. In particular it encodes the phenomenon of critical slowing down [6]—the fact that the equilibration time diverges as $T \rightarrow T_c$ —and governs the initial density of possible topological defects [7]. It also provides a link between fundamental physics and a commonly used phenomenological approach to out-of-equilibrium systems using Langevin equations [8–10] with potentially large qualitative impact on the behavior of such systems [11]. The fact that realistic systems will be prepared at some temperature and then cooled (or heated) “through” a phase transition during some finite time thus provides a strong motivation to study thermal field theory using the real-time formulation.

As for quantum field theory at vanishing temperature, there are in general no exact analytical methods to do calculations in finite temperature field theory. On the other hand, the methods we use in standard field theory may be more or less trivially adapted to systems in or close to thermal equilibrium. One may in particular tackle a number of nontrivial problems using standard perturbation theory, i.e. the expansion in some small coupling. It is however widely known that due to the emergence of an additional dimensionful parameter—namely the temperature T —the usual criteria for the (superficial) convergence of perturbation theory are modified. It turns out that even theories which are weakly coupled at $T=0$ may not be correctly described in perturbation theory close to a second or weakly first order phase

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transition. The reason is that at a critical point the correlation length of the order parameter field diverges and the theory is correlated over all scales. In such a situation the behavior of the perturbative series is drastically modified by infrared phenomena. Thus for example perturbation theory predicts a first order phase transition for $O(N)$ -symmetric scalar theories in $3+1$ dimension, whereas we know this transition to be of second order [12,13]. Also the damping rate mentioned above in perturbation theory is found to diverge at the phase transition, yielding “critical speeding up” instead of the expected slowing down.

A method which is well known in statistical physics and has become more widely spread in the particle theory community in recent years is the “Wilsonian renormalization group” invented specifically to deal with situations involving divergent correlation lengths [14]. The first application of this formalism to field theory is due to Polchinski ([15], see also [16]). The standard way of adapting the method for a relativistic field theory works in Euclidean space-time, and there are a number of concrete realizations of this approach. These range from the “environmentally friendly” RG ([17] and references therein) via the “auxiliary mass method” [18,19] to the “exact renormalization group” (ERG) of Wetterich [20]. These methods are straightforwardly adapted to finite temperature calculations in the Matsubara formalism. However, as pointed out above, there are a number of interesting quantities that may not be obtained in a direct way from calculations in the imaginary-time formalism and it thus is attractive to devise methods for calculations directly in real time that are also able to deal with critical theories. One such method is the “thermal renormalization group” (TRG), proposed by D’Attanasio and Pietroni [21] for scalar theories and later also generalized to gauge theories [22]. It was shown in [23] and [24] that this method provides a good tool to study the phase transition of (at vanishing temperature weakly coupled) scalar theories. Also a first calculation of the behavior of the damping rate of a scalar φ^4 theory close to its critical temperature was performed in [25] and critical slowing down was indeed observed.

In the present paper we will be concerned with the formulation of the TRG for theories with fermionic degrees of freedom. As an example we will study a simple model displaying a spontaneously broken chiral symmetry which is restored at high temperatures. This model involves two flavors of chiral fermions and an $O(4)$ -symmetric scalar sector with at $T=0$ spontaneously broken symmetry. A Yukawa coupling of the fermions to the order-parameter field then breaks the chiral symmetry in the presence of a nonvanishing vacuum expectation value (VEV). This modification of the sigma model was investigated in [26] and is believed to describe the $N_f=2$ chiral phase transition of QCD in the limit where the axial $U(1)$ symmetry is completely broken. Based on universality arguments the theory should undergo a phase transition of second order with critical exponents of an $O(4)$ -symmetric scalar model. After the formulation of the TRG we concentrate on the universal critical behavior. It turns out that the standard tool in the application of Wilsonian renormalization group equations, the derivative expansion, does not work in the fermionic sector, and we discuss

the implications of this observation for calculations in models with chiral symmetry. A necessary ingredient for fermion decoupling in the framework of the TRG is an effective mass term which does not break chiral symmetry and is obtained from considering “hard thermal loop” contributions [27]. These result in a complicated dispersion relation for fermionic fields and we discuss how the propagators may be simplified without losing the important effects.

The paper is organized as follows: In the next section we present the formulation of the thermal renormalization group equation for theories involving fermionic degrees of freedom. It will be a straightforward extension of the TRG for scalar theories and for details of the practical application including necessary approximations we refer the reader to [21,24]. In Sec. III we introduce the model we will use. Section IV contains the discussion of the fermionic dispersion relations and constitutes an important part of the present work. In Sec. V we then report on the results we find for the universal aspects of the critical behavior of the model under consideration. These compare well with results from other approaches and prove our method to constitute a good starting point for the nonperturbative exploration of dynamical aspects of the phase transition. Section VI gives a summary and conclusions.

II. THE THERMAL RENORMALIZATION GROUP FOR FERMIONS

In this section we discuss the formulation of the thermal renormalization group (TRG) for fermionic degrees of freedom. The TRG equation is a Wilsonian renormalization group equation in Minkowski space for the dependence of an effective action on an external cutoff scale Λ , where this cutoff is imposed only on the thermal fluctuations [21]. This is possible since in the real-time formulation of thermal field theory the propagators contain two parts, one being identical to the ($T=0$) propagator and another one being multiplied by the appropriate thermal distribution. We will impose a cutoff on this second part. A somewhat more extensive discussion of the closed time-path approach to thermal field theory in connection with Wilsonian renormalization group methods may be found in [24]. We will here assume that the reader is familiar with thermal field theory in real time, and only discuss the implementation of the cutoff and the corresponding RG equations.

As stated above, the suppression of soft contributions to the path integral is implemented by modifying the two-point functions. In the operator approach, the propagators are obtained as the expectation values of time-path-ordered products of two field operators at different space-time points. Introducing for fermionic fields the Fourier decomposition

$$\hat{\psi}(x) = \sum_{\pm s} \int \frac{d^3k}{(2\pi)^3 2\omega_k} [b(k,s)u(k,s)e^{(-ikx)} + d^\dagger(k,s)v(k,s)e^{(ikx)}]$$

$$\hat{\psi}^\dagger(x) = \sum_{\pm s} \int \frac{d^3k}{(2\pi)^3 2\omega_k} [b^\dagger(k, s) u^\dagger(k, s) e^{(ikx)} + d(k, s) v^\dagger(k, s) e^{(-ikx)}] \quad (1)$$

the propagator depends on the expectation values of products of b , $b^\dagger$, d , and $d^\dagger$. In thermal equilibrium at a given temperature $T = \beta^{-1}$ one has

$$\begin{aligned} \langle b^+(k, s) b(k', s') \rangle_\beta &= (2\pi)^3 2\omega_k \tilde{N}(\omega_k) \delta_{ss'} \delta(\vec{k} - \vec{k}') \\ \langle d^+(k, s) d(k', s') \rangle_\beta &= (2\pi)^3 2\omega_k \tilde{N}(\omega_k) \delta_{ss'} \delta(\vec{k} - \vec{k}') \\ \langle b(k, s) b^+(k', s') \rangle_\beta &= (2\pi)^3 2\omega_k (1 - \tilde{N}(\omega_k)) \delta_{ss'} \delta(\vec{k} - \vec{k}') \\ \langle d(k, s) d^+(k', s') \rangle_\beta &= (2\pi)^3 2\omega_k (1 - \tilde{N}(\omega_k)) \delta_{ss'} \delta(\vec{k} - \vec{k}') \end{aligned} \quad (2)$$

where $\tilde{N}(\omega)$ is the Fermi-Dirac distribution,

$$\tilde{N}(\omega) = \frac{1}{e^{\omega/T} + 1}. \quad (3)$$

The above expectation values differ from their bosonic counterparts only by the appearance of $\tilde{N}(\omega)$ instead of the Bose distribution $N(\omega)$ and by the minus-sign in the last two equations, this being due to the anticommuting nature of the fermion fields. The introduction of the cutoff scale Λ now proceeds in the same way as in the case of scalar fields [21], namely by replacing

$$\tilde{N}(\omega_k) \rightarrow \tilde{N}(\omega_k; \Lambda) \quad (4)$$

with

$$\tilde{N}(\omega_k; \Lambda) \rightarrow \begin{cases} \tilde{N}(\omega_k) & \text{for } |\vec{k}| \gg \Lambda, \\ 0 & \text{for } |\vec{k}| \ll \Lambda. \end{cases} \quad (5)$$

This implies that the “hard” modes are in equilibrium at temperature T whereas “soft” thermal fluctuations are suppressed. We will write

$$\tilde{N}(\omega_k; \Lambda) = \Theta(|\vec{k}|, \Lambda) \tilde{N}(\omega_k) \quad (6)$$

and in practical applications always use a sharp cutoff with $\Theta(|\vec{k}|, \Lambda) = \theta(|\vec{k}| - \Lambda)$. The cutoff thermal propagator then reads (α and β are spinor indices, T_c denotes time ordering on the contour C)

$$\begin{aligned} iS_{c,\Lambda}(x, x')_{\alpha\beta} &= \langle T_c \psi_\alpha(x) \bar{\psi}_\beta(x') \rangle_\beta^\Lambda \\ &= \theta_c(t - t') S_{c,\Lambda}^>(x, x')_{\alpha\beta} \\ &\quad + \theta_c(t' - t) S_{c,\Lambda}^<(x, x')_{\alpha\beta} \end{aligned} \quad (7)$$

with

$$\begin{aligned} S_{c,\Lambda}^>(x, x')_{\alpha\beta} &= \langle \psi_\alpha(x) \bar{\psi}_\beta(x') \rangle_\beta^\Lambda, \\ S_{c,\Lambda}^<(x, x')_{\alpha\beta} &= -\langle \bar{\psi}_\beta(x') \psi_\alpha(x') \rangle_\beta^\Lambda. \end{aligned} \quad (8)$$

Inserting Eqs. (1) and Eq. (2) with the modification Eq. (4) thus gives

$$\begin{aligned} iS_{c,\Lambda}(x - x')_{\alpha\beta} &= \int \frac{d^4k}{(2\pi)^4} 2\pi \delta(k^2 - m_0^2) \\ &\quad \times (\not{k} + m_0)_{\alpha\beta} e^{-ik(x-x')} \\ &\quad \times [\theta_c(t - t') \theta(k_0) + \theta_c(t' - t) \theta(-k_0) \\ &\quad - \tilde{N}(|k_0|) \Theta(|\vec{k}|, \Lambda)]. \end{aligned} \quad (9)$$

Now we may adopt the matrix notation which has become common in real-time thermal field theory (see [5]) and write the fermion propagator as

$$\begin{aligned} S_\Lambda(k) &= \begin{pmatrix} \Delta_0^f & (\Delta_0^f - \Delta_0^{f*}) \theta(-k_0) \\ (\Delta_0^f - \Delta_0^{f*}) \theta(k_0) & -\Delta_0^{f*} \end{pmatrix} \\ &\quad - (\Delta_0^f - \Delta_0^{f*}) \tilde{N}(|k_0|) \theta(|\vec{k}| - \Lambda) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \end{aligned} \quad (10)$$

with

$$\Delta_0^f = [\not{k} - m_0 + i\epsilon]^{-1}, \quad \Delta_0^{f*} = [\not{k} - m_0 - i\epsilon]^{-1}, \quad (11)$$

where we have already switched to a θ -function cutoff. Together with an analogous expression for the modified scalar propagator [21] we then define the scale dependent partition function $Z_\Lambda[J, \eta, \bar{\eta}]$ as

$$\begin{aligned} Z_\Lambda[J, \eta, \bar{\eta}] &= \int \delta\phi \delta\psi \delta\bar{\psi} \exp \left\{ i \left(\frac{1}{2} \phi \cdot D_\Lambda^{-1} \cdot \phi + \bar{\psi} \cdot S_\Lambda^{-1} \cdot \psi \right. \right. \\ &\quad \left. \left. + J \cdot \phi + \bar{\eta} \cdot \psi + \bar{\psi} \cdot \eta + S_{\text{int}}[\phi, \psi, \bar{\psi}] \right) \right\}. \end{aligned} \quad (12)$$

From the way the cutoff was introduced, it is obvious that

$$Z_{\Lambda \rightarrow \infty}[J, \eta, \bar{\eta}] = Z^{(T=0)}[J, \eta, \bar{\eta}] \quad (13)$$

and that in the opposite limit $\Lambda \rightarrow 0$ the usual generating functional for the theory at temperature T is obtained. Finally, since all scale dependence of Z_Λ is explicit and contained in the terms bilinear in the fields, it is straightforward to obtain an expression for the Λ dependence of Z_Λ .

As in the Euclidean formulation it is however more convenient to work with a slightly modified version of the effective action, the generating functional for the one particle irreducible (1PI) Green functions. This is obtained from the Legendre transform of the logarithm of Z_Λ according to

$$\begin{aligned} \Gamma_\Lambda[\varphi, \psi, \bar{\psi}] &= -i \ln Z_\Lambda - J \cdot \varphi - \bar{\eta} \cdot \psi - \bar{\psi} \cdot \eta \\ &\quad - \frac{1}{2} \varphi \cdot D_\Lambda^{-1} \cdot \varphi - \bar{\psi} \cdot S_\Lambda^{-1} \cdot \psi. \end{aligned} \quad (14)$$

Here the fields are now the “classical” fields as $\varphi = \delta(-i \ln Z_\Lambda)/\delta J$ etc. The sources on the right hand side are to be regarded as functionals of the fields, given by the inversion of the above expressions, and are thus scale dependent. The RG equation governing the scale dependence of Γ_Λ is now easily obtained and one finds

$$\Lambda \partial_\Lambda \Gamma_\Lambda[\Phi] = \frac{i}{2} \text{STr} \left\{ \left(\Lambda \partial_\Lambda \mathcal{D}_\Lambda^{-1} \right) \left(\frac{\delta^2 \Gamma_\Lambda}{\delta \Phi \delta \Phi} + \mathcal{D}_\Lambda^{-1} \right)^{-1} \right\} \quad (15)$$

which is a straightforward extension of the scalar TRG equations [21]). Φ denotes a superfield

$$\Phi = (\phi, \psi, \bar{\psi}), \quad \bar{\Phi} = (\phi, -\bar{\psi}, \psi) \quad (16)$$

and the supertrace takes into account the anticommuting nature of the Grassmann fields. Note that we have suppressed all indices: The fields carry a spacetime index, a thermal index, possible internal indices and (for the fermions) Dirac indices. All these are traced in Eq. (15). Also by our conventions the bare inverse propagator matrix reads

$$\mathcal{D}_\Lambda^{-1}(k) = \begin{pmatrix} D_\Lambda^{-1}(k) & & \\ & S_\Lambda^{-1}(k) & \\ & & (S_\Lambda^{-1}(k))^T \end{pmatrix} \quad (17)$$

and we have

$$\frac{\delta^2 \Gamma_\Lambda}{\delta \Phi \delta \Phi} \equiv \Gamma_\Lambda^{(2)} = \begin{pmatrix} \frac{\delta^2 \Gamma_\Lambda}{\delta \phi \delta \phi} & \frac{\delta^2 \Gamma_\Lambda}{\delta \phi \delta \psi} & \frac{\delta^2 \Gamma_\Lambda}{\delta \phi \delta \bar{\psi}} \\ -\frac{\delta^2 \Gamma_\Lambda}{\delta \bar{\psi} \delta \phi} & -\frac{\delta^2 \Gamma_\Lambda}{\delta \bar{\psi} \delta \psi} & -\frac{\delta^2 \Gamma_\Lambda}{\delta \bar{\psi} \delta \bar{\psi}} \\ \frac{\delta^2 \Gamma_\Lambda}{\delta \psi \delta \phi} & \frac{\delta^2 \Gamma_\Lambda}{\delta \psi \delta \psi} & \frac{\delta^2 \Gamma_\Lambda}{\delta \psi \delta \bar{\psi}} \end{pmatrix}. \quad (18)$$

The general properties of the TRG equation are extensively discussed in [21,24] and we will here only make a few remarks in order to make the present work self-contained. Let us first note that no approximations have been made at this point. The TRG equation (15) is an exact relation on the same level as for example the Schwinger-Dyson equations. However, it is of course impossible to give a general solution to this functional differential equation—note that the matrix $(\Gamma_\Lambda^{(2)} + \mathcal{D}_\Lambda^{-1})^{-1}$ is the full propagator. The equation is formally a one-loop equation and we may introduce the derivative $\tilde{\partial}_\Lambda$ which is defined to be a derivative acting only on the explicit scale dependence in the bare propagator. Using this derivative we have

$$\Lambda \partial_\Lambda \Gamma_\Lambda[\Phi] = \frac{i}{2} \text{STr} \Lambda \tilde{\partial}_\Lambda \ln(\Gamma_\Lambda^{(2)} + \mathcal{D}_\Lambda^{-1}) \quad (19)$$

and the one-loop character is apparent. We may thus give a simple recipe for calculations in the framework of the ther-

mal renormalization group: To derive the flow equation of a specific coupling, write down the one-loop diagrams that would contribute to this coupling, taking into account all vertices allowed by the symmetries of the theory under consideration. The flow equation is then obtained by considering only those parts of the diagram which involve thermal distribution functions (since the Λ dependence only comes with those in the TRG approach), replacing the distribution functions according to Eq. (4) and similarly for the Bose distribution, taking the derivative with respect to the explicit scale dependence and then performing the loop integral. It is obvious that this integral will be finite. The UV behavior of the thermal parts of any Feynman diagram is never problematic since any UV divergences of the full theory at finite temperature reside in the $(T=0)$ parts of the diagrams. The behavior in the infrared, which might be problematic in perturbation theory, is regulated by the introduction of the external scale [see Eq. (5)]. The formal exactness of this approach is due to the use of full propagators and vertices.

In order to simplify the calculations somewhat we may make use of an observation made in [28], namely the fact that one may get rid of the fields on the lower part of the CTP contour (the “thermal ghost fields”) if one is interested in physical observables. To this end one may introduce a functional $\bar{\Gamma}$ which depends on one (super)field Φ (without thermal indices) and is defined through

$$\bar{\Gamma}[\Phi] = \Gamma[\Phi_1, \Phi_2[\Phi_1]]|_{\Phi_1=\Phi}. \quad (20)$$

Here Φ_2 is the solution to its equation of motion for arbitrary Φ_1 , i.e. we have

$$\left. \frac{\delta \Gamma[\Phi_1, \Phi_2]}{\delta \Phi_2} \right|_{\Phi_2=\Phi_2[\Phi_1]} = 0. \quad (21)$$

For space-time independent fields one has $\Phi_2[\Phi_1] = \Phi_1$ and Eq. (20) is up to a constant equivalent to

$$\left. \frac{\delta \bar{\Gamma}[\Phi]}{\delta \Phi} \right|_{\Phi_1=\Phi_2=\Phi=\text{const}} = \left. \frac{\delta \Gamma[\Phi_1, \Phi_2]}{\delta \Phi_1} \right|_{\Phi_1=\Phi_2=\Phi=\text{const}}. \quad (22)$$

This is the definition that we will use below. The TRG equation for $\bar{\Gamma}_\Lambda^{(1)}$ [which is obtained from Eq. (22) after replacing Γ with Γ_Λ] is readily found from Eqs. (15) and (22). Some care is required in the presence of fermionic fields, since derivatives with respect to fermionic fields do not commute with a supermatrix like the full propagator. We have for example

$$\begin{aligned} \frac{\delta}{\delta \varphi} \Gamma_\Lambda^{(2)} &= \frac{\delta \Gamma_\Lambda^{(2)}}{\delta \varphi} + \Gamma_\Lambda^{(2)} \frac{\delta}{\delta \varphi} \\ \frac{\delta}{\delta \psi} \Gamma_\Lambda^{(2)} &= \frac{\delta \Gamma_\Lambda^{(2)}}{\delta \psi} + M \Gamma_\Lambda^{(2)} M \frac{\delta}{\delta \psi} \end{aligned} \quad (23)$$

where φ (ψ) is a bosonic (fermionic) field and $M = \text{diag}(1, -1, -1)$ (in the present case). The supertrace is cyclic for

supermatrices. Note also that the fermionic derivative of a supermatrix (e.g. $\delta\Gamma_\Lambda^{(2)}/\delta\psi$) is not a supermatrix.

In the following section we will first introduce the model that we will be interested in and then turn to a discussion of the approximations we have to make in order to convert the functional differential equation Eq. (15) into a solvable system of (partial) differential equations.

III. THE CHIRAL QUARK-MESON MODEL

We will be interested in a simple model involving only scalar and fermionic fields with scalar self-coupling as well as Yukawa couplings. It is well accepted that the chiral phase transition of QCD in the limit of two massless flavors and an explicitly broken axial $U(1)$ should be of second order, with $O(4)$ -model universal behavior. This could in principle be modeled by a simple $O(4)$ -symmetric scalar theory with (at $T=0$) spontaneously broken symmetry. Recently, Jungnickel and Wetterich have pointed out that a modification of the sigma model might actually be quantitatively connected to QCD by a combination of analytical and numerical calculations [26]. This model, the so-called “chiral quark-meson model,” includes [in the limit of strongly broken $U(1)_A$] the usual sigma-model fields, which are however coupled to N_c dublets of fermionic fields via a Yukawa coupling. As was pointed out above, perturbation theory is not applicable even for small ($T=0$) couplings at this phase transition. On the other hand, the Wilsonian RG was introduced to deal with such situations. The chiral quark-meson model is thus, apart from being interesting phenomenologically, an ideal testing ground for our method.

We shall not discuss in detail here how the model may be connected to QCD and refer the reader to [26] for details. We will actually in the present work not even try to make quantitative contact with this effective description by choosing the values of the ($T=0$) couplings accordingly—this could in principle be done and we briefly comment on this below. For the main part of this paper we shall discuss the universal physics obtained for this model in different approximations. These are independent of the initial values of the couplings and thus also apply to the $N_f=2$ chiral phase transition. Nonuniversal quantities will only be considered for small ($T=0$) couplings in the present work.

The initial condition that we will use for the effective action $\bar{\Gamma}_\Lambda$ as $\Lambda \rightarrow \infty$, i.e. the effective action of the ($T=0$) theory is [remember that the free two-point functions are subtracted in Γ_Λ , cf. Eq. (14)]

$$\bar{\Gamma}_{\Lambda \rightarrow \infty} = \int d^4x \left[-\frac{g_{(T=0)}}{2} \rho^2 - \bar{h}_{(T=0)} \bar{\psi} \left(\frac{\varphi + \varphi^\dagger}{2} + \gamma_5 \frac{\varphi - \varphi^\dagger}{2} \right) \psi \right]. \quad (24)$$

It involves a fermion flavor doublet as well as a scalar multiplet containing four real degrees of freedom,

$$\psi = (\psi^1, \psi^2), \quad \varphi = \frac{1}{2}(\sigma + i\vec{\pi}\vec{\tau}). \quad (25)$$

Furthermore we use the $O(4)$ symmetry to write

$$\rho = \text{Tr}(\varphi^\dagger \varphi) = \frac{1}{2}(\sigma^2 + \vec{\pi}^2). \quad (26)$$

The fermions come with an additional color index and closed fermion loops correspondingly yield a factor of N_c .

The effective action given in Eq. (24) is of course only a crude approximation to a full effective action of a ($T=0$) theory—one could e.g. include perturbative contributions. We will nevertheless use this simple form and report nonuniversal results only for small values of the initial couplings $g_{(T=0)}$ and $\bar{h}_{(T=0)}$, such that the higher order contributions should be small. The reason for doing so is that the universal behavior is unaffected by these approximations and that on the other hand the nonuniversal quantities would be most interesting using the ($T=0$) parameters that correspond to the realistic case where observables calculated from the model Eq. (24) match those found in nature. A discussion of this matching and the resulting values for the ($T=0$) couplings as obtained from Euclidean versions of Wilsonian renormalization group equations may be found in [26]. One finds for example that the Yukawa coupling is $\bar{h}^2 \sim \mathcal{O}(10)$ and thus by no means small. If we were interested in non-universal features of the chiral phase transition as obtained from the quark-meson model we would have to address the question of initial values carefully. Studies using the Matsubara formulation of thermal field theory in connection with the average effective action yield reasonable predictions for e.g. the critical temperature [29].

Within our approximations the effective action (24) thus constitutes the boundary value for the solution of the TRG equation (15) [combined with the definition of $\bar{\Gamma}_\Lambda$, Eq. (20)]. We now have to discuss the approximations we will use in order to bring this equation into a solvable form. The simplest approximation one may think of is to assume that the effective action at finite temperature always has the same form as given in Eq. (24) and that only the couplings—in this case the scalar mass term, the quartic coupling and the Yukawa coupling—depend on the scale Λ . One then obtains the flow equations of those couplings by plugging $\bar{\Gamma}_\Lambda$ from Eq. (24) into Eq. (15) and matching coefficients on both sides. This simple approximation will give completely misleading results, for reasons to be discussed in the next section. In the absence of fermions (for $\bar{h}=0$) this approximation would give a second order phase transition, however the values for the universal critical exponents will differ strongly from known results. The situation can be considerably improved if one allows for a general form of the effective action for a constant scalar field $\rho(x) = \rho$ [23,24]. The effective action for constant field is connected to the effective potential through

$$\bar{\Gamma}_\Lambda[\varphi = \sqrt{2\rho}, \psi = \bar{\psi} = 0] = (m^2\rho - U(\rho)) \int d^4x \quad (27)$$

for a constant configuration ρ with the bare mass m . The effective potential corresponds to the free energy for constant

fields and it approaches a universal form at a second order phase transition. For the results for this universal form in scalar $O(N)$ models, using the approximation of a standard kinetic term [as in Eq. (24)] and a general effective potential, see [24]. The critical exponents one finds in this case are in much better agreement with the known values than in the case of a simple polynomial approximation for U [21].

Neglecting corrections to the kinetic term may be motivated by arguing in the framework of an expansion of the effective action in powers of derivatives. In this expansion, the effective potential is obviously the lowest order term. The leading term beyond this approximation then is obtained from the scalar wave-function renormalization, i.e. using the approximation

$$\begin{aligned} \bar{\Gamma}_\Lambda[\varphi, \psi, \bar{\psi}] = & \int d^4x [(Z_\Lambda - 1) \text{Tr}(\partial^\mu \varphi^\dagger \partial_\mu \varphi) \\ & + m^2 \rho - U_\Lambda(\rho)] + \text{Fermion-contributions.} \end{aligned} \quad (28)$$

Here the wave-function renormalization Z_Λ , which could in principle be a function of the field, is evaluated at the minimum of the effective potential. This will be the approximation we use in the scalar sector of the model. It is widely used also in Euclidean versions of the Wilsonian renormalization group, see e.g. [29]. In particular we will neglect all imaginary contributions to the two-point functions. These would only indirectly affect the scale dependence of the effective potential through a contribution to the anomalous dimension. Since the anomalous dimension will turn out to be very small in all situations that we will be interested in, we believe that dropping imaginary contributions results in negligible errors in quantities like the critical temperature and the critical exponents (with the possible exception of η , see below). On the other hand the imaginary parts are interesting objects by themselves and we plan to study them in future work.

Let us now turn to the sector involving fermionic fields. Here the situation is more complicated. The reason for the complication is that in contrast to the bosonic sector, where the leading finite temperature effects show up in a local contribution to the mass $\propto T$ in lowest order derivative expansion, there is no local mass term generated for the fermions. We will come back to this issue in Sec. IV and for now only consider the interaction term. Here one obtains to lowest order in the derivative expansion a general function of the fields [which should respect the $SU(2) \times SU(2) \times U(1)$ symmetry of Eq. (24)]. However, one expects the fermions to decouple from the infrared physics and universal quantities should be independent of the approximations in the fermionic sector. The dependence of nonuniversal quantities as the critical temperature should in principle be calculable in perturbation theory and we may, as long as we restrict ourselves to small values of $\bar{h}_{(T=0)}$ and $\bar{g}_{(T=0)}$, always argue that any induced coupling involving fermions are of higher order in the $(T=0)$ couplings and thus negligible compared to the Yukawa coupling. We will thus approximate the interaction between fermions and bosons by the original form as

given in Eq. (24) for all values of Λ . In particular, we always evaluate the Yukawa coupling at $\rho=0$ and at vanishing energies and momenta.

Let us then discuss as an example the flow equation for the ρ derivative of the effective potential. This is obtained from the flow equation for the φ derivative of $\bar{\Gamma}_\Lambda$ at constant φ and vanishing $\bar{\psi}$ and ψ :

$$\begin{aligned} \Lambda \partial_\Lambda (\varphi U'(\rho)) &= \Lambda \partial_\Lambda \left. \frac{\delta \bar{\Gamma}_\Lambda}{\delta \varphi} \right|_{\varphi=\text{const}, \bar{\psi}=\psi=0} \\ &= \frac{i}{2} \text{STr} \left[\left. \mathcal{K}_\Lambda \frac{\delta^3 \bar{\Gamma}_\Lambda}{\delta \varphi \delta \Phi \delta \Phi} \right|_{\varphi=\text{const}, \bar{\psi}=\psi=0} \right]. \end{aligned} \quad (29)$$

Here we have introduced the kernel

$$\begin{aligned} \mathcal{K}_\Lambda &= - \left(\frac{\delta^2 \bar{\Gamma}_\Lambda}{\delta \Phi \delta \Phi} + \mathcal{D}_\Lambda^{-1} \right)^{-1} (\Lambda \partial_\Lambda \mathcal{D}_\Lambda^{-1}) \left(\frac{\delta^2 \bar{\Gamma}_\Lambda}{\delta \Phi \delta \Phi} + \mathcal{D}_\Lambda^{-1} \right)^{-1} \\ &= \begin{pmatrix} K_\Lambda^b & & \\ & K_\Lambda^f & \\ & & (K_\Lambda^f)^T \end{pmatrix}. \end{aligned} \quad (30)$$

The scalar kernel may be rewritten as [21]

$$K_\Lambda^b = -\Lambda \delta(|\vec{k}| - \Lambda) \Delta_\Lambda^b (\Delta_\Lambda^b)^* \frac{\Delta_0^b - (\Delta_0^b)^*}{\Delta_0^b (\Delta_0^b)^*} N(|k_0|) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \quad (31)$$

The fermion kernel is obtained from Eq. (31) by replacing the bare and full bosonic propagator Δ_0^b and Δ_Λ^b by the corresponding fermionic ones and substituting the Bose distribution $N(\omega)$ by the minus Fermi-Dirac one. Note that we have already inserted a sharp cutoff for Eq. (31). Because of the thermal matrix structure of the kernel and the fact that [28]

$$\frac{\delta^3 \bar{\Gamma}_\Lambda}{\delta \Phi \delta \Phi \delta \Phi} = \sum_{i,j=1,2} \frac{\delta^3}{\delta \Phi_i \delta \Phi_j \delta \Phi_1} \bar{\Gamma}_\Lambda \Big|_{\Phi_2=\Phi_1=\Phi} \quad (32)$$

the trace over the thermal indices in Eq. (29) allows us to rewrite the right hand side in terms of $\bar{\Gamma}_\Lambda$. Furthermore for real self-energies we may write

$$\Delta_\Lambda^b (\Delta_\Lambda^b)^* \frac{\Delta_0^b - (\Delta_0^b)^*}{\Delta_0^b (\Delta_0^b)^*} = 2\pi i \rho_\Lambda^b(q) \quad (33)$$

with the spectral density ρ_Λ and similarly for the fermions. The spectral densities may be obtained from the ansatz for the effective action $\bar{\Gamma}$. In our case, using the approximations discussed above for the scalar fields and for the moment dropping any momentum-dependent contribution other than the standard kinetic term for the fermions, we find

$$2\pi i \rho_{\Lambda}^b(q) = \frac{1}{Z_{\Lambda} q^2 - M^2 + i\epsilon} - \frac{1}{Z_{\Lambda} q^2 - M^2 - i\epsilon}$$

$$2\pi i \rho_{\Lambda}^f(q) = \frac{\not{q} + m_{\chi_{SB}}}{q^2 - m_{\chi_{SB}}^2 + i\epsilon} - \frac{\not{q} + m_{\chi_{SB}}}{q^2 - m_{\chi_{SB}}^2 - i\epsilon} \quad (34)$$

where $m_{\chi_{SB}}^2 = \bar{h}^2 \rho / 2$ and M^2 are the eigenvalues of the matrix $\partial^2 U / \partial \varphi \partial \varphi$. For $\epsilon \rightarrow 0$, the spectral densities yield δ functions which allow us to perform the q_0 integration implicit in the trace in Eq. (29). The angular integral is trivial for constant fields. Finally from the Λ derivative of the cut-off, we have another δ function [cf. Eq. (31)] which allows us to perform the integration with respect to the modulus of the loop momentum. The result for the flow equation of the effective potential is

$$\Lambda \partial_{\Lambda} U'(\rho) = -\frac{\Lambda^3}{4\pi^2} \left[(3U''(\rho) + 2\rho U^{(3)}(\rho)) \frac{N(\bar{\omega}_{\sigma})}{\sqrt{Z_{\Lambda} \bar{\omega}_{\sigma}}} \Theta(\bar{\omega}_{\sigma}^2) \right. \\ \left. + 3U''(\rho) \frac{N(\bar{\omega}_{\pi})}{\sqrt{Z_{\Lambda} \bar{\omega}_{\pi}}} \Theta(\bar{\omega}_{\pi}^2) \right] - N_c \frac{\Lambda^3}{\pi^2} \bar{h}^2 \frac{\tilde{N}(\bar{\omega}_{\psi})}{\bar{\omega}_{\psi}} \quad (35)$$

with

$$\bar{\omega}_{\sigma}^2 = Z_{\Lambda} \Lambda^2 + U'(\rho) + 2\rho U''(\rho)$$

$$\bar{\omega}_{\pi}^2 = Z_{\Lambda} \Lambda^2 + U'(\rho) \quad (36)$$

$$\bar{\omega}_{\psi}^2 = \Lambda^2 + \frac{\bar{h}^2 \rho}{2}.$$

The Θ functions arise from the bosonic spectral density [21] and will be understood in the following. The structure of Eq. (35) is again easy to understand from the one-loop character of the flow equation: the first contribution is from the radial mode which is massive at the minimum of $U(\rho)$, the second contribution is from the 3 Goldstone-bosons which are massless at the minimum, and the last contribution is from the $2N_c$ fermionic fields which are degenerate.

Similar flow equations may be obtained for the Yukawa coupling and the scalar wave-function renormalization constant. Before we come to an investigation of the critical behavior of the model using these flow equations in Sec. V, we will in the next section turn to the question of approximations to the fermionic two-point function.

IV. FERMION DECOUPLING, UNIVERSALITY AND HARD THERMAL LOOPS

In this section we will discuss how the fermions decouple from the critical behavior of the quark-meson model, leading to a second order phase transition with $O(4)$ -model critical behavior. We will argue that the mechanism of fermion decoupling at a second order phase transition is different from

the mechanism behind fermion decoupling at high temperatures.

Let us start with discussing the high-temperature limit of our model. We will work for the moment with the simplest approximation to the full flow equation for the effective action and only take into account the effective potential and the Yukawa coupling and set $Z_{\Lambda} = 1$. The flow equation for the Yukawa coupling reads within these approximations

$$\Lambda \partial_{\Lambda} \bar{h} = \frac{\Lambda^3}{4\pi^2} \bar{h}^3 \frac{1}{U'(0)} \left[\frac{N(\bar{\omega}_0)}{\bar{\omega}_0} + \frac{\tilde{N}(\Lambda)}{\Lambda} \right] \quad (37)$$

where

$$\bar{\omega}_0^2 = \Lambda^2 + U'(0). \quad (38)$$

Considering now the limit of T large compared to all scales in the theory we may use the asymptotic expressions

$$N(x) \xrightarrow{T \gg x} \frac{T}{x} \quad (39)$$

and

$$\tilde{N}(x) \xrightarrow{T \gg x} \frac{1}{2} \quad (40)$$

to obtain the system of flow equations (35) and (37) in the high-temperature limit

$$\Lambda \partial_{\Lambda} U'(\rho)|_{\text{high-T}} = -\frac{T\Lambda^3}{4\pi^2} \left[\frac{3U''(\rho) + 2\rho U^{(3)}(\rho)}{\Lambda^2 + U'(\rho) + 2\rho U''(\rho)} \right. \\ \left. + 3 \frac{U''(\rho)}{\Lambda^2 + U'(\rho)} \right] + \dots \quad (41)$$

$$\Lambda \partial_{\Lambda} \bar{h}|_{\text{high-T}} = \frac{T\Lambda^3}{4\pi^2} \bar{h}^3 \frac{1}{U'(0)(\Lambda^2 + U'(0))} + \dots$$

The contribution of the fermions to the flow of the effective potential thus vanishes in the high-temperature limit, due to the different behavior of the distribution functions (39) and (40). This is what is referred to as “fermion decoupling” and the TRG thus reproduces the fact that the dynamics of the soft modes in the high-temperature limit is independent of the fermionic content of the theory.

Let us now turn to the question about fermion decoupling at a second order phase transition. One is interested in this case in a (three-dimensional) scaling solution for the free energy (i.e. the effective potential). In order to investigate this scaling solution it is useful to introduce rescaled variables according to [23]

$$\kappa = \frac{\rho}{\Lambda T}, \quad u(\kappa) = \frac{U(\rho)}{\Lambda^3 T}, \quad u^{(n)} = \Lambda^{n-3} T^{n-1} U^{(n)}. \quad (42)$$

At this point we do not rescale the Yukawa coupling, since if we assume the fermions to decouple there is no reason why this coupling should approach a scaling solution. Introducing furthermore the dimensionless flow parameter

$$\lambda = \frac{\Lambda}{T} \quad (43)$$

we obtain for the flow of $u'(\kappa)$ from Eq. (35)

$$\begin{aligned} \lambda \partial_\lambda u'(\kappa) = & -2u'(\kappa) + \kappa u''(\kappa) \\ & - \frac{\lambda}{4\pi^2} \left[(3u'' + 2\kappa u^{(3)}) \frac{n(\lambda\omega_\sigma)}{\omega_\sigma} + 3u'' \frac{n(\lambda\omega_\pi)}{\omega_\pi} \right] \\ & - N_c \frac{1}{\pi^2} \bar{h}^2 \frac{\tilde{n}(\lambda\omega_\psi)}{\omega_\psi} \end{aligned} \quad (44)$$

with the dimensionless quantities

$$\begin{aligned} \omega_\sigma^2 &= 1 + u'(\kappa) + 2\kappa u''(\kappa) \\ \omega_\pi^2 &= 1 + u'(\kappa) \\ \omega_\psi^2 &= 1 + \frac{\bar{h}^2 \kappa}{2\lambda} \end{aligned} \quad (45)$$

and $n(x) = [\exp(x) - 1]^{-1}$, $\tilde{n}(x) = [\exp(x) + 1]^{-1}$. The flow equation for the Yukawa coupling is rewritten accordingly as

$$\lambda \partial_\lambda \bar{h} = \bar{h}^3 \frac{1}{4\pi^2} \frac{1}{u'(0)} \left[\frac{n(\lambda\omega_0)}{\omega_0} + \tilde{n}(1) \right] \quad (46)$$

where $\omega_0 = 1 + u'(0)$. We will see that the problems with decoupling within the present approximation concern the region of small κ , thus to simplify the discussion and to make our point clear we expand the effective potential in powers of κ . Introducing couplings according to

$$u(\kappa) = \sum_n \frac{u_n}{n!} \kappa^n \quad (47)$$

we obtain flow equations for the u_n from Eq. (44). Let us for example consider the scale dependence of u_1 , which is directly obtained from Eq. (44) by setting $\kappa = 0$. It involves the scalar mass term u_1 itself as well as the scalar four-point coupling u_2 (remember $\rho = \varphi^2/2$) and the Yukawa coupling $\bar{h}$. A scaling solution for the scalar couplings u_n may arise only in the three-dimensional limit, i.e. for $\lambda \rightarrow 0$. In order to identify the corresponding fixed-point values of the couplings we thus expand the flow equations. In the case of u_1 we have

$$\lambda \partial_\lambda u_1 = -2u_1 - \frac{3}{2\pi^2} \frac{u_2}{1+u_1} - N_c \frac{1}{2\pi^2} \bar{h}^2 + \mathcal{O}(\lambda) \quad (48)$$

and the fermionic contributions obviously do not decouple—a possible fixed-point value would depend on $\bar{h}$

and N_c , in contradiction with the assumption of universality. It is not hard to see what spoils the high-temperature argumentation in the present case: It is the fact that we should really not compare the behavior of the distribution functions alone, but rather the contribution of couplings times the distribution functions as $\lambda \rightarrow 0$. Then it is clear that for a scaling solution (i.e. at a second order phase transition) while the scalar distribution function diverges for small λ as λ^{-1} , the four scalar coupling multiplying it is renormalized to 0 with the same power of λ . The product remains finite for $\lambda \rightarrow 0$. In contrast, the fermion distribution function does not diverge, but on the other hand the Yukawa coupling is not driven to 0 with λ —the product also remains finite and is thus for all λ of the same order as the scalar contribution.

This is a particular feature of the critical point. For $T \neq T_c$, away from the second order phase transition, the scalar coupling is not renormalized to zero and indeed the product of coupling times distribution function for large T/ω dominates the fermionic contribution as shown above.

Also notice that the presence of a fermion mass term (as for $\kappa \neq 0$) changes the situation. While the behavior of coupling times distribution function does not change ($\tilde{n}$ still approaches a constant), now the factor ω_ψ^{-1} in the fermion contribution to $\lambda \partial_\lambda u'(\kappa)$ behaves as $\sqrt{2\lambda/\bar{h}^2 \kappa}$ and suppresses the fermion contribution. This observation will help us to see how universality is saved and fermion decoupling works, a question to which we turn now.

To this end, let us turn again to the truncation used to derive Eqs. (35) and (37). We have pointed out that we have neglected all terms beyond the effective potential and the Yukawa coupling. In a purely scalar theory, it is known that using only the effective potential one already obtains quite reasonable results for the critical behavior (see e.g. [24]). This is due to the fact that for the critical theory one is effectively using an expansion in the (small) anomalous dimension. There is however one other important aspect of the thermal effects on scalar fields which is fully covered by the approximation with only an effective potential—the generation of a thermal mass. One may easily check that within the present approximation one reproduces the “hard thermal loop” (HTL) result for the scalar mass term: Consider the present model for simplicity with $N_c = 0$. The flow equation for the mass then reads from above

$$\begin{aligned} \Lambda \partial_\Lambda U'(\rho) = & - \frac{\Lambda^3}{4\pi^2} \left[(3U''(\rho) + 2\rho U^{(3)}(\rho)) \frac{N(\bar{\omega}_\sigma)}{\omega_\sigma} \right. \\ & \left. + 3U''(\rho) \frac{N(\bar{\omega}_\pi)}{\omega_\pi} \right]. \end{aligned} \quad (49)$$

Adopting a perturbative point of view, neglecting $U^{(3)}(\rho)$ and replacing in the spirit of the HTL approximation $\bar{\omega}_\sigma = \bar{\omega}_\pi = \Lambda$, we obtain

$$\partial_\Lambda U'(\rho) = -6 \frac{\Lambda}{4\pi^2} U''(\rho) \frac{1}{e^{\Lambda/T} - 1}. \quad (50)$$

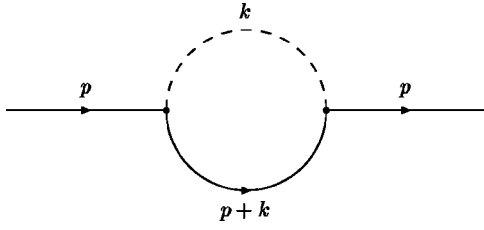


FIG. 1. The one-loop contribution to the fermion self-energy.

Remembering that the boundary condition for the flow is the $(T=0)$ -effective action and neglecting—again to leading order in perturbation theory—the scale dependence of $U''(\rho)$ we easily solve the flow equation (50) to find the temperature-dependent mass term to be given by

$$U'(\rho, T) = U'(\rho, T=0) + \frac{U''(\rho)}{4} T^2 \quad (51)$$

reproducing the leading order perturbative result. The positive contribution $\propto T^2$ of course is also the reason for the restoration of the spontaneously broken symmetry. The reason why this leading order high-temperature result is reproduced in the simple truncation with only the effective potential is that it is obtained as a local contribution to the effective action. Of course, this local contribution is fully resummed if we solve the flow equation (35) without further approximations.

In contrast to this simple behavior of the leading thermal contributions in the scalar case, for the fermionic fields there is no local mass term generated by hard thermal loops. It is however well known that the leading order corrections to the fermion propagator lead to modified dispersion relations which for small momenta resemble the ones of a massive particle without breaking chiral invariance for $\kappa=0$ [5]. If we neglect thermal contributions to the momentum dependence of the fermion two-point function, or in fact for any truncation that assumes analyticity for the momentum dependence of the fermion two-point function, we will miss this important finite temperature contribution. The effect is numerically small in the presence of a large enough chiral symmetry breaking mass, but in the situation discussed above with $\kappa=0$, this effective mass term takes care of the decoupling of the fermionic fluctuations, as we will argue in the following.

Let us first briefly review the structure of the leading contribution to the fermion self-energy for large T in perturbation theory [5]. The contributing diagram is given in Fig. 1. In the framework of the hard thermal loop approximation one then neglects the external momentum squared, p^2 , as well as the masses of the particles in the loop compared to the loop momentum squared. After these simplifications the integrations may be performed and one obtains for the two-point function of a massless fermion ($P = \sqrt{p^2}$)

$$(\Delta^f)^{-1}(p) = (1 + a(p_0, P))\not{p} + b(p_0, P)\not{u}$$

$$a(p_0, P) = \frac{m_T^2}{P^2} \left[1 - \frac{p_0}{2P} \ln \left(\frac{p_0 + P}{p_0 - P} \right) \right] \quad (52)$$

$$b(p_0, P) = \frac{m_T^2}{P} \left[-\frac{p_0}{P} + \frac{1}{2} \left(\frac{p_0^2}{P^2} - 1 \right) \ln \left(\frac{p_0 + P}{p_0 - P} \right) \right]$$

where m_T^2 is a quantity with the dimension of $(\text{mass})^2$, given by

$$m_T^2 = \frac{\hbar^2}{16} T^2. \quad (53)$$

The dispersion relation corresponding to Eq. (52) is given by [we use $u=(1,0,0,0)$ for the rest frame of the heat bath as is commonly done]

$$0 = D(\omega) = [1 + a(\omega, P)]^2 (\omega^2 - P^2) + 2[1 + a(\omega, P)]b(\omega, P)\omega + b(\omega, P)^2. \quad (54)$$

This equation has four solutions $\pm \omega_{\pm}$ which can however only be given analytically in limiting cases. A general feature is that the solutions $\omega = \pm \omega_{+}$ have the same (positive) ratio of helicity over chirality and are thus connected to the usual solution at vanishing temperature. The other solutions, $\omega = \pm \omega_{-}$ have a negative helicity over chirality ratio and constitute a collective mode termed the “plasmino.” As $P \rightarrow 0$, both solutions approach each other and

$$\omega_{\pm}(P) \xrightarrow{m_T \gg P} m_T \pm \frac{P}{3}. \quad (55)$$

In the opposite limit, for $P \gg m_T$, one finds that $\omega_{-}(P)$ exponentially approaches P with an also exponentially suppressed residue, whereas $\omega_{+}(P) \rightarrow P + m_T^2/P$. The most important observation is however that the complicated momentum-dependence of the effective fermion propagator as given in Eq. (52) gives rise to a dispersion relation which resembles for small momenta P that of a massive particle with a mass $\propto T$, while chiral symmetry is not broken by this behavior. It is also clear that the result given in Eq. (52) may not be reproduced using a derivative expansion as is routinely done in calculations using variants of the Wilsonian renormalization group. Since on the other hand in the framework of the TRG it is exactly this behavior that guarantees fermion decoupling, we need to find a way to incorporate the above results in an approximation scheme for the flow equations. We now turn to this question.

The canonical way to incorporate nontrivial momentum dependence of n -point functions in the renormalization-group approach would be to expand the effective action in powers of the fields rather than in powers of derivatives. In order to obtain a closed system of flow equations one then has to truncate the infinite tower of n -point functions by hand, either by setting the n -point functions with n larger than some n_{max} to zero, or by using e.g. perturbative expres-

sions for those. We could in principle follow such an approach, using scalar n -point functions derived from an effective action as given above and not imposing any restrictions on the momentum dependence of the fermion two-point function. Since however the scale dependence of the fermion self-energy is given by a flow equation which involves a loop integration over a fermion propagator (it is basically given by the Λ derivative of the diagram in Fig. 1 where the two-point functions are replaced by the modified two-point functions as introduced in Sec. II), we would end up with a rather complicated integro-differential equation.

On the other hand, we may also use the knowledge about the structure of the leading thermal corrections to the fermion propagator as discussed above. This structure is not changed by the presence of an infrared-cutoff scale Λ . All dependence of the leading correction on Λ is in the thermal mass term (53), which becomes a function of Λ . Thus instead of (52), the thermal fermion two-point function for finite values of Λ reads to leading order in T , taking into account a possible chiral symmetry-breaking mass $m_{\chi SB}$ now,

$$(\Delta^f)^{-1}(p; \Lambda) = (1 + a(p_0, P; \Lambda))\not{p} + b(p_0, P; \Lambda)\not{t} - m_{\chi SB}$$

$$a(p_0, P; \Lambda) = \frac{m_{T, \Lambda}^2}{P^2} \left[1 - \frac{p_0}{2P} \ln \left(\frac{p_0 + P}{p_0 - P} \right) \right] \quad (56)$$

$$b(p_0, P; \Lambda) = \frac{m_{T, \Lambda}^2}{P} \left[-\frac{p_0}{P} + \frac{1}{2} \left(\frac{p_0^2}{P^2} - 1 \right) \ln \left(\frac{p_0 + P}{p_0 - P} \right) \right].$$

Correspondingly, the dispersion relation is obtained from Eq. (54) by replacing m_T by $m_{T, \Lambda}$ and subtracting $m_{\chi SB}^2$. The solutions have the same general features as discussed above. If we use this approximation for the fermion propagator in the flow equations for the effective potential and the Yukawa coupling, we find the following result:

$$\begin{aligned} \Lambda \partial_\Lambda U'(\rho) = & -\frac{\Lambda^3}{4\pi^2} \left[(3U''(\rho) + 2\rho U^{(3)}(\rho)) \frac{N(\bar{\omega}_\sigma)}{\bar{\omega}_\sigma} \right. \\ & \left. + 3U''(\rho) \frac{N(\bar{\omega}_\pi)}{\bar{\omega}_\pi} \right] \\ & - 2N_c \frac{\Lambda^3}{\pi^2} \bar{h}^2 \left[\frac{\tilde{N}(\bar{\omega}_+)}{D'(\bar{\omega}_+)} - \frac{\tilde{N}(\bar{\omega}_-)}{D'(\bar{\omega}_-)} \right] \quad (57) \end{aligned}$$

and

$$\begin{aligned} \Lambda \partial_\Lambda \bar{h} = & -\frac{\Lambda^3}{4\pi^2} \bar{h}^3 \left\{ \frac{N(\bar{\omega}_0)}{\bar{\omega}_0 D(\bar{\omega}_0)} + 2 \left[\frac{\tilde{N}(\bar{\omega}_{+,0})}{D'(\bar{\omega}_{+,0})(\bar{\omega}_{+,0}^2 - \bar{\omega}_0^2)} \right. \right. \\ & \left. \left. - \frac{\tilde{N}(\bar{\omega}_{-,0})}{D'(\bar{\omega}_{-,0})(\bar{\omega}_{-,0}^2 - \bar{\omega}_0^2)} \right] \right\}. \quad (58) \end{aligned}$$

Here $\bar{\omega}_\sigma$, $\bar{\omega}_\pi$, and $\bar{\omega}_0$ have been given in Eqs. (36) and (38), whereas $\bar{\omega}_\pm$ and $\bar{\omega}_{\pm,0}$ are the positive solutions of

$$\begin{aligned} 0 = D(\omega) = & [1 + a(\omega, \Lambda; \Lambda)]^2 (\omega^2 - \Lambda^2) \\ & + 2[1 + a(\omega, \Lambda; \Lambda)]b(\omega, \Lambda; \Lambda)\omega \\ & + b(\omega, \Lambda; \Lambda)^2 - \frac{\bar{h}^2 \rho}{2} \quad (59) \end{aligned}$$

at $\rho \neq 0$ and $\rho = 0$ respectively. Also primes on $D(\omega)$ denote the derivative of $D(\omega)$ from Eq. (59) with respect to ω . We now obtain an additional flow equation which governs the dependence of the thermal mass $m_{T, \Lambda}$ [appearing in a and b above; see Eq. (56)] on Λ . This flow equation reads

$$\begin{aligned} \Lambda \partial_\Lambda m_{T, \Lambda}^2 = & -\bar{h}^2 \frac{\Lambda^3}{4\pi^2} \left\{ \frac{N(\bar{\omega}_0)}{\bar{\omega}_0} + 2 \left[\frac{\tilde{N}(\bar{\omega}_{+,0})}{D'(\bar{\omega}_{+,0})} \right. \right. \\ & \left. \left. - \frac{\tilde{N}(\bar{\omega}_{-,0})}{D'(\bar{\omega}_{-,0})} \right] \right\}. \quad (60) \end{aligned}$$

In this way we again have a closed system of differential equations which furthermore—in contrast to Eqs. (35) and (37), being obtained in lowest order derivative expansion—contains the most important corrections induced by thermal fluctuations. Even though the boundary value for the thermal mass term $m_{T, \Lambda}^2$ for $\Lambda \rightarrow \infty$ is zero, by Eq. (60) for any finite $\Lambda \geq 0$ and $\bar{h}^2$ one gets a nonvanishing effective thermal mass for the fermions even for $\rho = 0$. In this way, remembering the discussion above, fermion decoupling is guaranteed also at the second order phase transition, since again the “energy-denominator” measured in units of Λ diverges as $\Lambda \rightarrow 0$.

We could thus stop the discussion at this point and use Eqs. (57)–(60) to discuss the critical behavior of our model. However, we shall now show that following a proposal made in [30] (in a different context), one may furthermore simplify the system of flow equations without introducing large errors in interesting quantities. This further simplification will turn out to be very useful once we aim to go beyond the lowest order in the derivative expansion in the scalar sector.

The key observation is that one may greatly simplify the effective fermion propagator without changing the most important features by writing

$$\Delta^f(p) = \frac{(1 + a(p_0, P))\not{p} + b(p_0, P)\not{u} + m_{\chi_{SB}}}{(1 + a(p_0, P))^2 p^2 + 2(1 + a(p_0, P))b(p_0, P)p_0 + b(p_0, P)^2 - m_{\chi_{SB}}^2} = \frac{\not{p} + m_{\chi_{SB}} + \mathcal{O}(m_T^2/P)}{p^2 - 2m_T^2 - m_{\chi_{SB}}^2 + \mathcal{O}(m_T^4/P^2)} \quad (61)$$

and neglecting terms suppressed by powers of P . Clearly, for $m_{\chi_{SB}}=0$, the resulting expression preserves chiral invariance on the one hand, but displays the behavior of a massive propagator as far as the dispersion relation is concerned on the other hand. If we use this modification, the flow equations for the potential, the Yukawa coupling, and the thermal mass read

$$\begin{aligned} \Lambda \partial_\Lambda U'(\rho) &= -\frac{\Lambda^3}{4\pi^2} \left[(3U''(\rho) + 2\rho U^{(3)}(\rho)) \frac{N(\bar{\omega}_\sigma)}{\bar{\omega}_\sigma} \right. \\ &\quad \left. + 3U''(\rho) \frac{N(\bar{\omega}_\pi)}{\bar{\omega}_\pi} \right] - N_c \frac{\Lambda^3}{\pi^2} \bar{h}^2 \frac{\tilde{N}(\bar{\omega}_\psi)}{\bar{\omega}_\psi} \\ \Lambda \partial_\Lambda \bar{h} &= \frac{\Lambda^3}{4\pi^2} \bar{h}^3 \frac{1}{U'(0) - 2m_{T,\Lambda}^2} \left[\frac{N(\bar{\omega}_0)}{\bar{\omega}_0} + \frac{\tilde{N}(\bar{\omega}_{\psi,0})}{\bar{\omega}_{\psi,0}} \right] \end{aligned} \quad (62)$$

and

$$\Lambda \partial_\Lambda m_{T,\Lambda}^2 = -\bar{h}^2 \frac{\Lambda^3}{4\pi^2} \left[\frac{N(\bar{\omega}_0)}{\bar{\omega}_0} + \frac{\tilde{N}(\bar{\omega}_{\psi,0})}{\bar{\omega}_{\psi,0}} \right] \quad (63)$$

with $\bar{\omega}_\psi^2 = \Lambda^2 + 2m_{T,\Lambda}^2 + m_{\chi_{SB}}^2$, $\bar{\omega}_{\psi,0}^2 = \Lambda^2 + 2m_{T,\Lambda}^2$. Since the fermions decouple from the universal critical behavior both in the simplified version of the flow equations [(62) and (63)] as well as in the full HTL-improved versions (57)–(60), all

universal results will be independent of the simplifications performed in order to arrive at Eq. (62).

Before we turn to a discussion of the universal critical behavior in the next section, let us consider the modifications brought about by the emergence of a thermal mass for the fermions and the quality of the approximation (61). In order to check the universality of the critical behavior we may consider the model for different values of the Yukawa coupling $\bar{h}_{(T=0)}$ at the respective critical temperature. If we approach the critical temperature from below, the effective potential always has a nontrivial minimum at a finite value of $\kappa = \kappa_0$. If we consider the rescaled field and couplings according to Eq. (41), we expect the value of κ_0 to be independent of λ for small values of the flow parameter. In Fig. 2 we display the dependence of the rescaled minimum of the potential on $t = \ln \lambda$ at the critical temperature for various initial values of the Yukawa coupling (unless stated otherwise, we always choose $g_{(T=0)} = 0.1$ and vary $x = \bar{h}_{(T=0)}^2/g_{(T=0)}$). A scaling solution is reached for $t \lesssim -7$ and one nicely observes the independence of the final value $\kappa_{0,*}$ of x . It is remarkable that the fermion contribution is sizable down to $t < 0$, i.e. $\Lambda < T$.

In order to discuss the importance of contributions neglected according to Eq. (61) we display in Fig. 3 the value of the thermal fermion mass $m_{T,\Lambda}^2$ divided by Λ (all dimensional quantities are given in units of $\sqrt{\rho_{(T=0)}}$). This is the relevant quantity in applications of Eq. (61) in the framework of the TRG with a sharp cutoff, since the three-

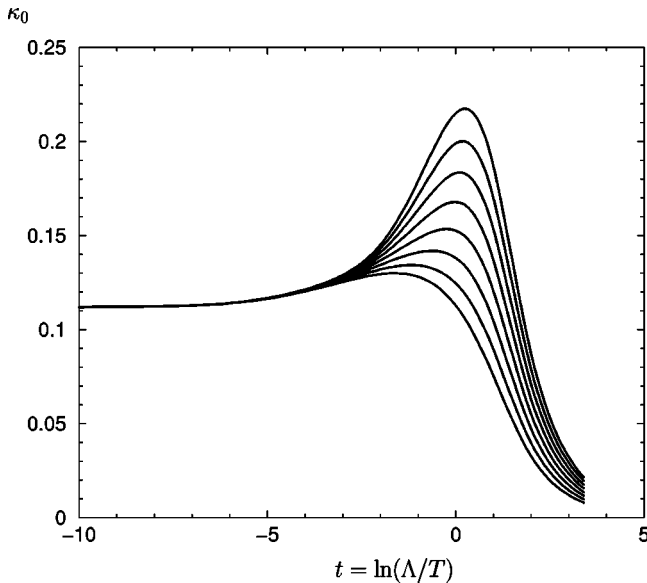


FIG. 2. The dimensionless minimum κ_0 as a function of t at the critical temperature for various values of $x = \bar{h}_{(T=0)}^2/g_{(T=0)}$ ($x = 0.01, 0.2, 0.4, 0.6, 0.8, 1, 1.2, 1.4$ from lowest to uppermost curve).

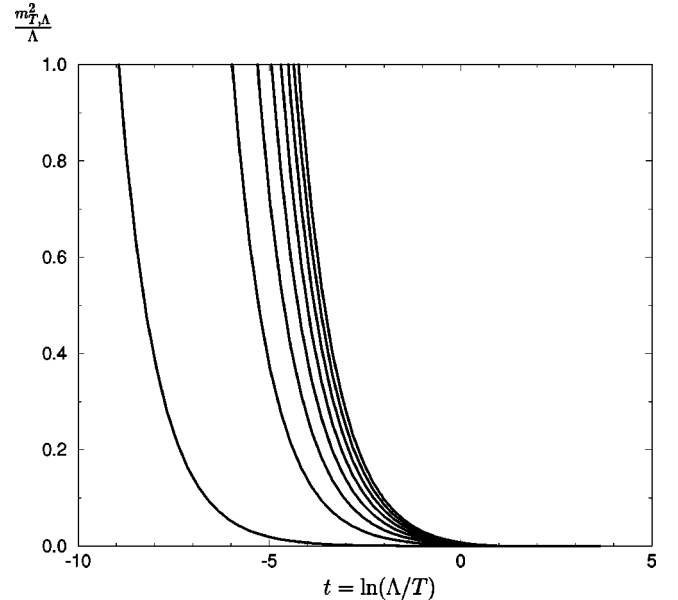


FIG. 3. The value of the thermal mass $m_{T,\Lambda}^2$ divided by Λ as a function of t at the critical temperature for various values of x ($x = 0.01, 0.2, 0.4, 0.6, 0.8, 1, 1.2, 1.4$ from leftmost to rightmost curve).

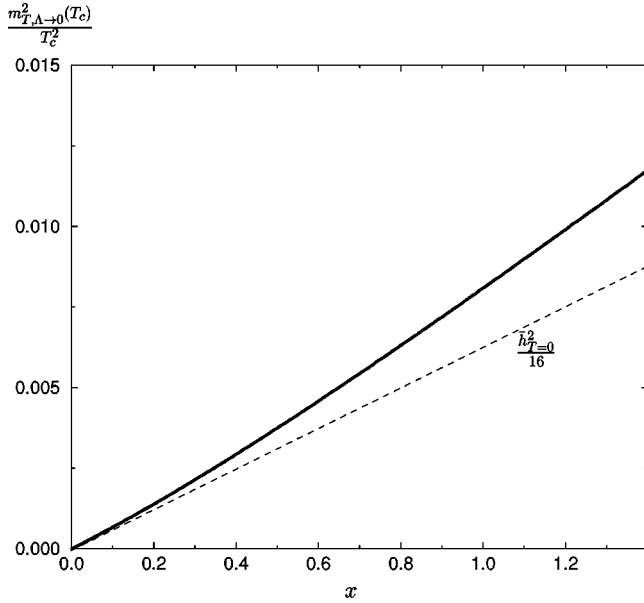


FIG. 4. The thermal mass $m_{T,\Lambda \rightarrow 0}^2$ in units of T^2 at the critical temperature as a function of x (solid line). For comparison the leading order result from Eq. (53) is also given (dashed line).

momenta inside the loop contributing to the flow of any quantity are $P=\Lambda$ due to the δ function in the kernel [Eq. (31) and correspondingly for the fermionic degrees of freedom]. We plot in Fig. 3 $m_{T,\Lambda}^2/\Lambda$ again as a function of t for the same values of $g_{(T=0)}$ and x as used in Fig. 2. We note that the thermal mass remains very small for large t —a one-loop calculation for finite Λ yields

$$\begin{aligned}
 m_{T,\Lambda}^2 &= \int_{\Lambda}^{\infty} dx \bar{h}^2 \frac{x}{4\pi^2} \left(\frac{1}{e^{x/T}-1} + \frac{1}{e^{x/T}+1} \right) \\
 &= \frac{\bar{h}^2}{12} T^2 + \frac{\bar{h}^2}{4\pi^2} T [T \operatorname{dilog}(e^{\Lambda/T}) \\
 &\quad + T \operatorname{dilog}(e^{\Lambda/T}+1) + \Lambda \ln(e^{\Lambda/T}+1)] \\
 &\approx \frac{\bar{h}^2}{2\pi^2} T e^{-\Lambda/T} (\Lambda + T) \quad (\Lambda/T \text{ large}) \quad (64)
 \end{aligned}$$

and $m_{T,\Lambda}^2/\Lambda$ is smaller than 0.2 for the values of x considered here for all $t \gtrsim -2.5$. On the other hand, as is obvious from Fig. 2), the main contribution from the fermionic degrees of freedom is at larger values of t . Thus we believe that the simplification of the fermionic dispersion relation made here introduces only very small errors for the nonuniversal results—as was pointed out above, universal quantities are completely unaffected.¹

¹Also note that neglected contributions in Eq. (61) are formally of higher order in $\bar{h}^2$ through Eq. (64).

Before turning to the phase transition itself, consider finally the value of $m_{T,\Lambda}^2$ as $\Lambda \rightarrow 0$. This is displayed in Fig. 4, where we have plotted $m_{T,\Lambda \rightarrow 0}^2/T^2$ at the critical temperature as a function of x again. For comparison the one-loop result (53) is displayed as dashed line. The renormalization of the Yukawa coupling through thermal fluctuations yields a small but non-negligible deviation of this quantity from its lowest-order value. We point out that here the HTL contribution [in the fermionic sector approximated by Eq. (61)] is consistently resummed.

V. THE UNIVERSAL CRITICAL BEHAVIOR AT THE CHIRAL PHASE TRANSITION

In this section we finally discuss the results for the chiral phase transition as obtained from the quark-meson model after including the leading thermal effects in the fermionic sector as discussed above. We have pointed out that the derivative expansion fails for the fermion propagator and that fermion decoupling at a second order phase transition is only guaranteed by the nonlocal fermion mass term produced by the thermal fluctuations. After taking this effect into account, we expect the phase transition to be in the universality class of the $O(4)$ model. Critical exponents and amplitude ratios should be independent of the number of colors or the Yukawa coupling. Before we turn to the scalar sector and the universal quantities, let us briefly discuss two nonuniversal aspects: the critical temperature and the behavior of the Yukawa coupling.

In this section we introduce an additional effect, namely we go to first order in the derivative expansion in the scalar sector by taking into account a scalar wave-function renormalization. This quantity was already included in $\bar{\Gamma}_{\Lambda}$, Eq. (28). Whereas Z_{Λ} is in principle a function of the order-parameter field, we evaluate it at the minimum of the effective potential. Its t dependence is given by

$$\frac{\partial Z_{\Lambda}}{\partial t} = -\eta Z_{\Lambda} \quad (65)$$

with the anomalous dimension η . Having introduced Z_{Λ} we consider instead of the dimensionless quantities from Eq. (42) renormalized dimensionless couplings according to

$$\kappa_r = Z_{\Lambda} \kappa, \quad h^2 = Z_{\Lambda}^{-1} \bar{h}^2, \quad u_r^{(n)} = Z_{\Lambda}^{-n} u^{(n)}. \quad (66)$$

We evaluate the anomalous dimension by considering the flow equation for the π -two-point function and expanding it around $\vec{q}^2=0$. Since the heat bath introduces an additional four-vector u_{μ} , there are in principle two different wave-function renormalizations for [choosing as usually $u_{\mu}=(1,0,0,0)$] $\vec{q}$ and q_0 . We do not make this distinction and consider only the effect of Z_{Λ} connected to $\vec{q}^2$, since this is the quantity which through Eq. (65) yields the critical exponent η (see below). For the anomalous dimension we find the following expression:

$$\begin{aligned}
\eta = & \lambda \frac{\kappa_r (u_r^{(2)})^2}{12\pi^2} (F(\omega_\sigma, \omega_\pi) + F(\omega_\pi, \omega_\sigma))|_{\kappa_r = \kappa_{r,0}} + N_c \frac{h^2}{12\pi^2 \omega_\psi^7} \left\{ [6\omega_\psi^4 - 3\omega_\psi^2 - 9\mu_T^2 \omega_\psi^2 + 10\mu_T^2](-\tilde{n}(\lambda \omega_\psi)) \right. \\
& + \lambda \omega_\psi \tilde{n}(\lambda \omega_\psi) [\tilde{n}(\lambda \omega_\psi) - 1] + [\omega_\psi^2 + 3\omega_\psi^2 \mu_T^2 - 4\mu_T^2] \lambda^2 \omega_\psi^2 \tilde{n}(\lambda \omega_\psi) [\tilde{n}(\lambda \omega_\psi) - 1] [2\tilde{n}(\lambda \omega_\psi) - 1] \\
& \left. + \frac{2}{3} \mu_T^2 \lambda^3 \omega_\psi^3 \tilde{n}(\lambda \omega_\psi) [\tilde{n}(\lambda \omega_\psi) - 1] [6\tilde{n}(\lambda \omega_\psi)^2 - 6\tilde{n}(\lambda \omega_\psi) + 1] \right\} \quad (67)
\end{aligned}$$

with

$$\begin{aligned}
F(\omega_\sigma, \omega_\pi) = & \frac{1}{\omega_\sigma \omega_\pi^5 (\omega_\sigma^2 - \omega_\pi^2)^3} \{ n(\lambda \omega_\sigma) (6\omega_\pi^7 - 8\omega_\pi^5 - 6\omega_\sigma^2 \omega_\pi^5) + n(\lambda \omega_\pi) (3\omega_\sigma^5 - 3\omega_\sigma^5 \omega_\pi^2 - 10\omega_\sigma^3 \omega_\pi^2) \\
& + 12\omega_\sigma^3 \omega_\pi^4 + 15\omega_\sigma \omega_\pi^4 - 9\omega_\sigma \omega_\pi^6 - \lambda n(\lambda \omega_\pi) [n(\lambda \omega_\pi) + 1] (3\omega_\sigma^5 \omega_\pi^3 - 3\omega_\sigma^5 \omega_\pi + 10\omega_\sigma^3 \omega_\pi^3) \\
& - 6\omega_\sigma^3 \omega_\pi^5 + 3\omega_\sigma \omega_\pi^7 - 7\omega_\sigma \omega_\pi^5) + \lambda^2 n(\lambda \omega_\pi) [n(\lambda \omega_\pi) + 1] [2n(\lambda \omega_\pi) + 1] (\omega_\sigma^5 \omega_\pi^2 - 2\omega_\sigma^3 \omega_\pi^4 + \omega_\sigma \omega_\pi^6) \} \quad (68)
\end{aligned}$$

where $\mu_T^2 = m_{T,\Lambda}^2 / \Lambda^2$, $\mu_\chi^2 = h^2 \kappa_{r,0} / 2\lambda$, $\omega_\psi^2 = 1 + 2\mu_T^2 + \mu_\chi^2$ and ω_σ as well as ω_π are defined in Eq. (45) [where the renormalized quantities according to Eq. (66) should be used]. In the limit where Λ is small compared to all other scales in the theory (this is the limit relevant for the fixed-point behavior) this rather complicated expression reduces to

$$\eta = \frac{\kappa_r (u_r^{(2)})^2}{6\pi^2} \frac{4\omega_\sigma^4 - 3\omega_\sigma^4 \omega_\pi^2 - 3\omega_\sigma^2 \omega_\pi^4 + 4\omega_\pi^4}{\omega_\sigma^6 \omega_\pi^6} \Big|_{\kappa_r = \kappa_{r,0}} + \mathcal{O}(\lambda) \quad (69)$$

and we again note the decoupling of the fermionic degrees of freedom in this limit.

The flow equations for the renormalized couplings introduced in Eq. (66) receive additional contributions proportional the anomalous dimension. We thus have for example

$$\begin{aligned}
\partial_t h = & \frac{\eta}{2} h + \frac{h^3}{4\pi^2 u_r^{(1)}(0)} \left[\frac{n(\lambda \omega_{r,0})}{\omega_{r,0}} + \tilde{n}(1) \right] \\
\partial_t u_r^{(1)}(\kappa_r) = & (-2 + \eta) u_r^{(1)} + (1 + \eta) \kappa_r u_r^{(2)} \\
& - \frac{\lambda}{4\pi^2} \left[(3u_r^{(2)} + 2\kappa_r u_r^{(3)}) \frac{n(\lambda \omega_\sigma)}{\omega_\sigma} + 3u_r^{(2)} \frac{n(\lambda \omega_\pi)}{\omega_\pi} \right] - \frac{N_c}{\pi^2} h^2 \frac{\tilde{n}(\lambda \omega_\psi)}{\omega_\psi} \quad (70)
\end{aligned}$$

and for the renormalized dimensionless minimum

$$\partial_t \kappa_{r,0} = -(1 + \eta) \kappa_{r,0} + \frac{\lambda}{4\pi^2} \left[\left(3 + 2\kappa_r \frac{u_r^{(3)}}{u_r^{(2)}} \right) \frac{n(\lambda \omega_\sigma)}{\omega_\sigma} + 3 \frac{n(\lambda \omega_\pi)}{\omega_\pi} \right] \Big|_{\kappa_r = \kappa_{r,0}} + \frac{N_c}{\pi^2} \frac{h^2}{u_r^{(2)}} \frac{\tilde{n}(\lambda \omega_\psi)}{\omega_\psi} \Big|_{\kappa_r = \kappa_{r,0}}. \quad (71)$$

Since the value of the anomalous dimension at the critical temperature will turn out to be very small, the modifications of the results displayed in Figs. 3 and 4 are tiny.

The value of the unrenormalized Yukawa coupling $\bar{h}^2$ at T_c as $\Lambda \rightarrow 0$ is displayed as a function of x in Fig. 5. Also plotted is the ($T=0$) Yukawa coupling and we note that it is only slightly modified through thermal fluctuations (for small values of the initial couplings). Here the introduction of a renormalized Yukawa coupling however has important consequences: The value of η is positive at the critical temperature. This leads to a vanishing renormalized Yukawa coupling at T_c , since in the infrared the flow equation for h^2 reduces to

$$\partial_t h^2 = \eta h^2 \quad (T = T_c, \Lambda \rightarrow 0). \quad (72)$$

Since η is very small, h^2 only vanishes very slowly ($h^2 \propto \lambda^\eta$).

Let us now turn to the critical temperature. In leading order perturbation theory, one obtains T_c as a function of $x = \bar{h}_{(T=0)}^2 / g_{(T=0)}$ as

$$T_c^{\text{pert}} = \sqrt{\frac{4\rho_0(T=0)}{1+x}}. \quad (73)$$

This dependence on x is basically reproduced by our resummed calculation. We display the critical temperature in

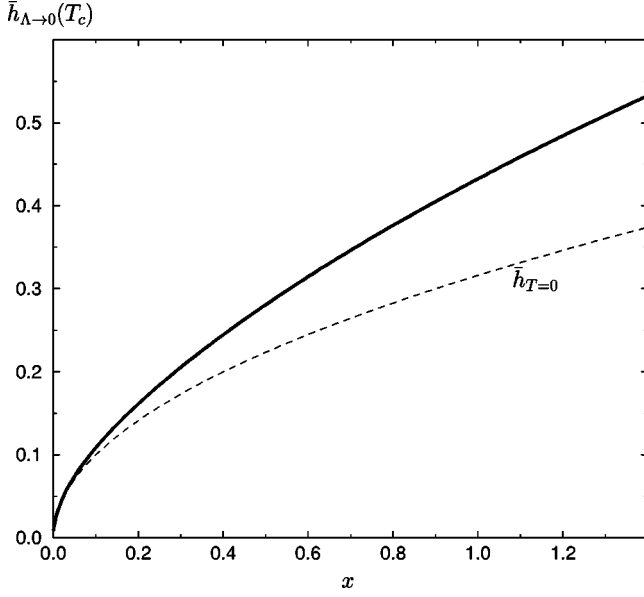


FIG. 5. The Yukawa coupling $\bar{h}_{\Lambda \rightarrow 0}(T_c)$ as a function of x (solid line). For comparison $\bar{h}_{T=0}$ is also given (dashed line).

units of $\sqrt{\rho_0(T=0)}$ as a function of x in Fig. 6. Note that in the limit $x \rightarrow 0$ the result does not approach Eq. (73). This is of course due to the fact that this limit corresponds to a pure scalar theory where there are still modifications of T_c^{pert} [24]. Indeed, the relative difference of our result with lowest order perturbation theory as displayed in Fig. 7 is small but non-vanishing for $x \rightarrow 0$.

To end the discussion, let us report the results for the universal behavior. As the universal quantities of the $O(N)$ models are well studied for various values of N (for a review see [31]) and the cases $N=1$ and $N=4$ have been studied with the methods used here in [21,23,24] we will only briefly

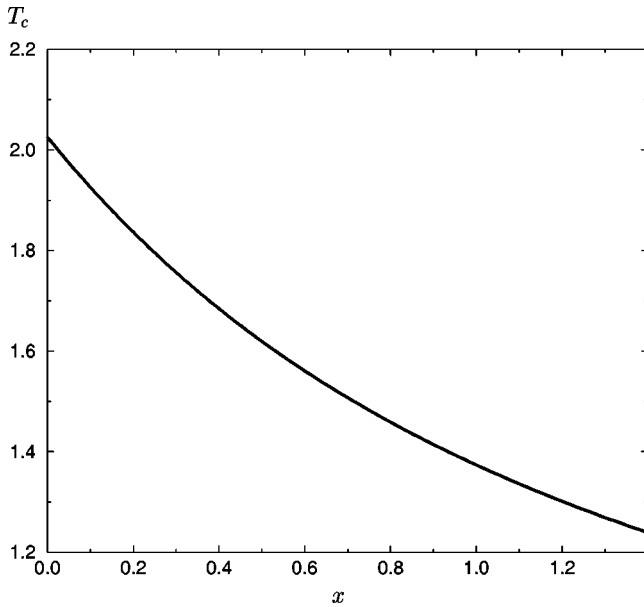


FIG. 6. The critical temperature in units of $\sqrt{\rho_0(T=0)}$ as a function of $x = \bar{h}_{T=0}^2/g_{(T=0)}$ (we use $g_{(T=0)}=0.1$ here).

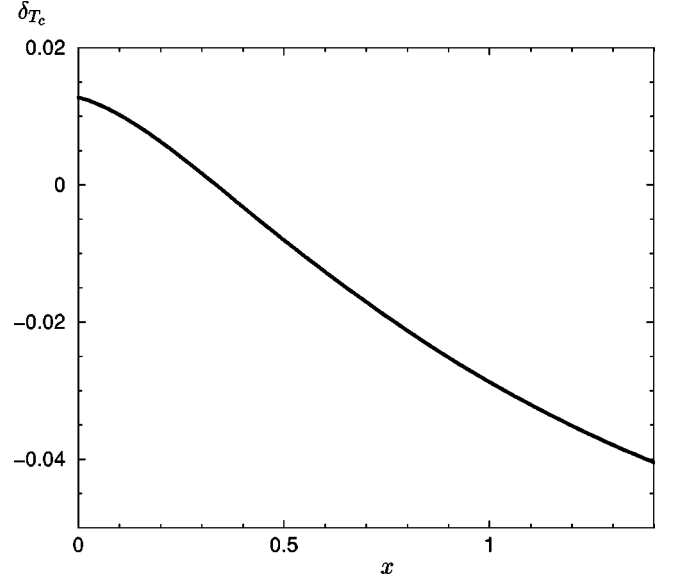


FIG. 7. The relative deviation of the critical temperature from the perturbative result, $\delta_{T_c} = 2[(T_c - T_c^{\text{pert}})/(T_c + T_c^{\text{pert}})]$ as a function of x ($g_{(T=0)}=0.1$).

discuss the main differences of our results with the ones that may be found in the literature.

The universal critical behavior is encoded in the scaling form of the equation of state, which relates the order parameter ρ , the temperature T and the external field $H = \sqrt{2\rho}U'(\rho)$ and may for example be written as [31]

$$\frac{\partial U}{\partial \rho} = (\sqrt{2\rho})^{\delta-1} f(x), \quad x = \frac{T - T_c}{T} (\sqrt{2\rho})^{-1/\beta} \quad (74)$$

with the Widom scaling function $f(x)$ and critical exponents δ and β . The Widom scaling function for the $O(4)$ model was calculated in the framework of the thermal renormalization group in [24], where however the anomalous dimension was neglected. It has also been computed from the exact renormalization group in the Matsubara formalism [29], from three-dimensional lattice simulations [32], and from mean-field theory as well as in second order ϵ expansion [33]. The TRG results have been compared to those of the exact renormalization group in [24] and good agreement was found apart from the behavior for large x and $T > T_c$. The disagreement in that region was due to differences in the value of the critical exponent γ , which governs the behavior of $f(x)$ as $x \rightarrow \infty$. In [24] it was argued that these differences should mostly be due to the neglected effects of the anomalous dimension. We are now in a position to check this claim through a calculation of critical exponents taking into account η according to Eq. (67).

We have calculated the exponents β and δ as introduced in Eq. (74), η as given in Eq. (67), as well as γ and ν . The exponents γ and ν encode the behavior of the unrenormalized and the renormalized mass of the order parameter field as $T \rightarrow T_c$ according to

TABLE I. Critical exponents for the $O(4)$ model.

	β	γ	ν	δ	η
This work	0.429	1.68	0.85	4.90	0.017
TRG + LO DE [24]	0.433	1.73	0.86	5.0	
ERG [29]	0.407	1.548	0.787	4.80	0.0344
3d PT [35]	0.38	1.44	0.73	4.82	0.03
3d MC [36]	0.384	1.48	0.748	4.85	0.025

$$\begin{aligned} \frac{m_r(T)}{T_c} &\propto \left(\frac{T - T_c}{T_c} \right)^\nu \\ \frac{m(T)}{T_c} &\propto \left(\frac{T - T_c}{T_c} \right)^{\gamma/2} \end{aligned} \quad (75)$$

whereas η describes the behavior of the two-point function of the order-parameter field at the critical temperature. Our results for these exponents are given in Table I, where we also give the results obtained from other methods for comparison.²

A useful consistency check is provided by the scaling relations which connect the critical exponents and leave only two of them independent. Thus for example one has [31])

$$\begin{aligned} \delta &= \frac{d+2-\eta}{d-2+\eta} \\ \gamma &= \nu(2-\eta) \\ \gamma &= \beta(\delta-1) \\ \beta &= \frac{\nu}{2}(d-2+\eta). \end{aligned} \quad (76)$$

These scaling relations are fulfilled by our results to better than 0.5%.

The results from [35] are obtained from a fixed-dimension calculation in the three-dimensional $O(4)$ model to seven loops whereas the results given in the last line are from three-dimensional Monte Carlo simulations of the $O(4)$ model [36]. We note that the results obtained in the present work, derived using the next-to-leading contributions in the derivative expansion in the scalar sector, are somewhat closer to the values found in lattice and three-dimensional perturbation theory calculations than the results given in [24]. However, the improvement for the exponents β , γ and ν is perhaps surprisingly small for the present model.

At this point it is interesting to compare the situation with the results from a study of the pure scalar $O(1)$ model. In [37] a sharp-cutoff Wilsonian RG was used to obtain critical exponents of the three-dimensional $O(1)$ model. We have explicitly verified that our flow equation for the effective

potential and the anomalous dimension, adapted to the $O(1)$ scalar theory³ reproduce the expression given there in the scaling limit. The truncations made in [37] are then identical to the ones made here and we would reproduce their values for the exponents. In that case, the exponents from the Wilsonian RG are in very good agreement with the results from other methods. This shows that in the $O(1)$ model, the next-to-leading order derivative expansion as used here gives accurate results as one would expect given the smallness of the anomalous dimension.⁴ The fact that the agreement in the present case is not as good might be due to the existence of a contribution in the present order of the derivative expansion which exists for $N > 1$ and has been neglected here, namely a term of the form $\tilde{Z}_\Lambda \varphi_a \varphi_a \partial^2 \varphi_b \varphi_b$ (see also [24]). This question remains to be settled in future work.

VI. SUMMARY AND CONCLUSIONS

In the present work we have extended the formulation of the thermal renormalization group [21] to theories involving fermionic degrees of freedom. Whereas the adaption of the full flow equation (Sec. II) is very much straightforward and does not involve any conceptually new points, it turns out that in performing approximations to the full functional differential equation one has to be considerably more careful for fermionic than for bosonic fields. After describing in Sec. III the specific model for the $N_f=2$ chiral phase transition that we studied, we have discussed at length questions concerning the decoupling of the fermionic fields at the phase transition in Sec. IV.

The model under consideration consists of an $O(4)$ -symmetric scalar field coupled via Yukawa couplings to $2N_c$ chiral fermions. At low temperatures the $O(4)$ symmetry is spontaneously broken down to an $O(3)$, where correspondingly 3 massless scalar fields (“pions”) appear. The massive scalar field is the would-be “sigma,” the fermionic fields acquire masses and the chiral symmetry thus is spontaneously broken. This model is expected to be in the universality class of the three-dimensional 4-vector model with a second order phase transition with well known universal behavior. The fermionic fields should not play a role for the infrared behavior at the phase transition and the effective potential should display scaling behavior for small field values.

We have then pointed out that the usual arguments about fermion decoupling at high temperatures do not apply in the TRG approach in the presence of second or weakly first order phase transitions. The fact that the scalar distribution function dominates the fermionic one for small energies (or large temperatures) does not directly imply a dominance of

²For results from the ERG see also [34] where results that are essentially identical to those given in [29] were obtained from a polynomial approximation to the effective potential.

³This is trivial for the flow equation for the potential. The anomalous dimension has to be replaced by the one defined from the sigma since there are of course no pions in the $O(1)$ case. We have calculated the corresponding expression also in the present model.

⁴As a caveat, one should however notice that the scaling relations (76) are violated by up to $\sim 6\%$ for the values given in [37].

the scalar contributions to the renormalized couplings (or the effective potential). The reason for this is the fact that it is not the distribution functions alone that enter the calculations, but the product of couplings times distribution functions. In the presence of a scaling solution for the effective potential, the couplings are also scale dependent in such a way that the product of scalar self couplings times scalar distribution functions has the same infrared behavior as the fermionic contribution, since in the latter the couplings do not exhibit scaling behavior.

This is in contrast to the high-temperature limit, where neither the scalar nor the Yukawa couplings scale. In this case it is indeed the relative behavior of the distribution functions that matters and fermion decoupling in the limit $T \rightarrow \infty$ works by the usual arguments.

One of the main points of the present work was then made in Sec. IV where we have discussed the question how universality in the presence of chiral fermions is nevertheless obtained and how this effect can be successfully implemented within the thermal renormalization group approach. The key observation is that even in chirally symmetric theories thermal fluctuations modify the fermionic dispersion relations such that the corrected dispersion relation resembles that of a massive particle for small momenta [27]. This is the leading effect of finite temperature in the fermionic sector of the theory and is in this sense on equal footing with the scalar mass correction $\propto T^2$ which drives the phase transition. Due to the chiral nature of the fermions, the thermal mass in the fermionic sector is however necessarily an effective mass resulting from a complicated momentum dependence of the two-point function. We have pointed out that this effect is not included if one applies a derivative expansion to the fermion two-point function as is generally done in the application of Wilsonian renormalization group equations. While universal quantities are unaffected, the presence of the effective thermal fermion mass should modify nonuniversal quantities as obtained within the derivative expansion [29].

We also discussed in Sec. IV how one may approximate the fermionic dispersion relations such that the resulting expressions are simple enough to keep the system of flow equa-

tions manageable on the one hand and not lose the important effects on the other hand. Here we follow a proposal put forward in [30] to rewrite the fermion propagator such that the thermal mass term is effectively local, but chiral symmetry remains unbroken by it. We argue that this approximation is quantitatively reliable for small ($T=0$) couplings.

After having clarified how fermion decoupling works within the TRG we have finally discussed the results of a study of the chiral phase transition in the framework of the quark-meson model in Sec. V. Indeed, the phase transition is second order with universal critical behavior as obtained from the pure scalar $O(4)$ model in three dimensions. We find that for small enough ($T=0$) couplings the critical temperature is rather well described by perturbation theory. This observation was already made in [13,24,38] for pure scalar $O(N)$ models. We improved upon previous results from the TRG on the critical exponents by including a scalar wavefunction renormalization, i.e. going beyond leading order in the derivative expansion in the scalar sector. The critical exponents are displayed in Table I. We have also discussed the quality of our approximations as far as the critical exponents are concerned. It turns out that the inclusion of the anomalous dimension has a rather small effect on the exponents β , γ and ν .

In the present paper we have been concerned only with quantities which may also be obtained from Wilsonian renormalization group calculations in the Matsubara formalism. In principle, the formalism set up here is directly applicable to quantities like damping rates, etc. [25]. After we have clarified the necessary ingredients in the fermionic sector, we are now in a position to study out-of-equilibrium properties of the present model in linear response theory.

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